



PERGAMON

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Chaos, Solitons and Fractals 17 (2003) 907–919

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

How to excite the internal modes of sine-Gordon solitons

J.A. González^{a,*}, A. Bellorín^{b,1}, L.E. Guerrero^c

^a *Laboratorio de Física Computacional, Centro de Física, Instituto Venezolano de Investigaciones Científicas, Apartado Postal 21827, Caracas 1020-A, Venezuela*

^b *Laboratorio de Física Teórica de Sólidos, Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Apartado Postal 47586, Caracas 1041-A, Venezuela*

^c *Grupo de la Materia Condensada, Departamento de Física, Universidad Simón Bolívar, Apartado Postal 89000, Caracas 1080-A, Venezuela*

Accepted 20 November 2002

Abstract

We investigate the dynamics of the sine-Gordon solitons perturbed by spatiotemporal external forces. We prove the existence of internal (shape) modes of sine-Gordon solitons when they are in the presence of inhomogeneous space-dependent external forces, provided some conditions (for these forces) hold. Additional periodic time-dependent forces can sustain oscillations of the soliton width. We show that, in some cases, the internal mode even can become unstable, causing the soliton to decay in an antisoliton and two solitons. In general, in the presence of spatiotemporal forces the soliton behaves as a deformable (non-rigid) object. A soliton moving in an array of inhomogeneities can also present sustained oscillations of its width. There are very important phenomena (like the soliton–antisoliton collisions) where the existence of internal modes plays a crucial role. We show that, under some conditions, the dynamics of the soliton shape modes can be chaotic. A short report of some of our results has been published in [Phys. Rev. E 65 (2002) 065601(R)].

© 2003 Elsevier Science Ltd. All rights reserved.

1. Introduction

The sine-Gordon solitons are very important in physics. They possess crucial applications in both particle physics and condensed matter theory. For instance, in solid state physics, they describe domain walls in ferromagnets, dislocations in crystals, charge density waves, fluxons in long Josephson junctions and Josephson transmission lines, etc. [1–6].

In general, non-integrable soliton equations (e.g. the ϕ^4 equation and the double sine-Gordon) may possess internal degrees of freedom, which are crucial in many phenomena [7–10]. A recent discussion of internal modes of solitary waves can be found in Ref. [11].

In some cases the internal modes can appear due to the discretization of the continuum equations [12–14].

However, it is well-known that the unperturbed (“pure”), continuum sine-Gordon equation does not have internal modes.

A very remarkable question is the following: *can external forces create internal modes in the sine-Gordon equation?*

Recently there has been a hot debate in the scientific literature about the existence of internal modes of sine-Gordon solitons.

* Corresponding author. Fax: +58-2-504-1148.

E-mail address: jorge@pion.ivic.ve (J.A. González).

¹ A. Bellorín would like to thank CDCH-UCV for support under project PI-03-11-4647-2000.

Some authors [15–21] have claimed that they have found an internal quasimode described as a long-lived oscillation of the width of the sine-Gordon soliton.

On the other hand, a very recent and interesting paper is contradicting all these reports [22].

By considering the response of the soliton to ac forces and initial distortions, Quintero et al. [22] show that neither intrinsic internal modes nor “quasimodes” exist in contrast to previous reports. We should stress that they use only time-dependent perturbations in their work.

In the present paper we study the sine-Gordon equation perturbed by *spatiotemporal* external forces:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F(x, t). \quad (1)$$

We will show that with some spatially inhomogeneous forces, the internal modes can exist for the sine-Gordon equation.

The paper is organized as follows: In Section 2 we investigate the existence and stability of the internal modes of sine-Gordon solitons in the presence of space-dependent external forces. In Section 3 we present the results of numerical experiments which confirm the theoretical predictions. In Section 4 we study the dynamics of the soliton in the presence of a space-dependent force that creates a double-well potential for the soliton. Additionally, the soliton is perturbed by a driving spatiotemporal force. Section 5 is dedicated to the case where the soliton is moving in an array of inhomogeneities.

2. Spatially inhomogeneous external forces

We have shown in previous papers [23–29] that in equations as the following:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F_1(x), \quad (2)$$

if the force $F_1(x)$ possesses a zero at x^* ($F_1(x^*) = 0$), this can be an equilibrium position for the soliton. If there is only one zero, this is a stable equilibrium position for the soliton if $(\partial F_1(x)/\partial x)_{x=x^*} > 0$. For the antisoliton, it is stable if $(\partial F_1(x)/\partial x)_{x^*} < 0$.

Let us assume that $F_1(x)$ is defined as:

$$F_1(x) = 2(B^2 - 1) \sinh(Bx) / \cosh^2(Bx). \quad (3)$$

This is a function with a zero at $x^* = 0$.

We have chosen this function because it possesses the following properties: (i) the exact solution for the soliton resting on the equilibrium position can be obtained, and (ii) the stability problem of this soliton can be solved exactly. The results obtained with this function can be generalized qualitatively to other systems topologically equivalent to this one. Besides, real physical systems are related to this example [3]. For instance, in a Josephson junction a perturbation that can be described by a function of type $F(x) = dR(x)/dx$, where $R(x)$ is a bell-shaped function, is an Abrikosov vortex lying in the junction's plane perpendicular to its local dimension [30].

A similar function can describe a local deformation of a charge density wave system [31].

Usually, function $R(x)$ is taken as the Dirac's δ -function. However, if we wish to model a finite-width inhomogeneity, function (3) is a better choice.

The exact stationary solution of Eq. (2), with $F_1(x)$ as defined in (3), is the following:

$$\phi_k = 4 \arctan[\exp(Bx)]. \quad (4)$$

The stability analysis, which considers small amplitude oscillations around ϕ_k [$\phi(k, x) = \phi_k(x) + f(x)e^{i\lambda t}$], leads to the eigenvalue problem [23–29]:

$$\widehat{L}f = \Gamma f, \quad (5)$$

where $\widehat{L} = -\partial_x^2 + [1 - 2 \cosh^{-2}(Bx)]$ and $\Gamma = -\lambda^2 - \gamma\lambda$.

This problem can be solved exactly [32]. The eigenvalues of the discrete spectrum [23–29] are given by the formula

$$\Gamma_n = B^2(\Lambda + 2An - n^2) - 1, \quad (6)$$

where $\Lambda(\Lambda + 1) = 2/B^2$.

The integer part of Λ (i.e. $[\Lambda]$) yields the number of eigenvalues in the discrete spectrum (i.e. $n \leq \Lambda$), which correspond to the soliton modes: this includes the translational mode Γ_0 , and the internal or shape modes Γ_n with $n > 0$ [23–29].

All this theoretical investigation produces the following results (note that parameter B will be our control parameter):

For $B^2 > 1$, the translational mode is stable and there are no internal modes.

If $1/3 < B^2 < 1$, then the translational mode is unstable. However, still there are no internal modes.

When $1/6 < B^2 < 1/3$, apart of the translational mode, there is one internal mode! This internal mode is stable.

In the case that $B^2 < 1/6$ there can appear many other internal modes! The exact number is $[A] - 1$, where $A(A + 1) = 2/B^2$.

For $B^2 < 2/\Lambda_*(\Lambda_* + 1)$, where $\Lambda_* = (5 + \sqrt{17})/2$, the first internal mode becomes unstable!

3. Cauchy problem—numerical experiments

What happens when we shift the soliton center-of-mass away from the equilibrium position?

We have the following initial problem:

$$\phi(x, 0) = 4 \arctan \{ \exp[B(x - x_0)] \}, \tag{7}$$

$$\phi_t(x, 0) = 0. \tag{8}$$

In the stable case ($B^2 > 1$) the center-of-mass of the soliton will make damped oscillations (for $x_0 \neq 0$) around the equilibrium point $x^* = 0$ (see Fig. 1).

In the case that the translational mode is unstable ($1/3 < B^2 < 1$), the soliton will move away indefinitely from the equilibrium position (see Fig. 2).

Consider the next initial problem:

$$\phi(x, 0) = 4 \arctan[\exp(Bx)] + C \sinh(Bx) \cosh^{-A}(Bx), \tag{9}$$

$$\phi_t(x, 0) = 0. \tag{10}$$

In this initial problem the initial soliton is deformed.

For $1/6 < B^2 < 1/3$ we will observe oscillations of the soliton width (see Fig. 3). This is due to the fact that an internal mode ($n = 1$) has been excited. Eventually, due to unavoidable errors in the initial conditions or to energy exchange between the internal mode and the translational mode, the soliton will move away from the equilibrium position (remember that the equilibrium position is unstable for the soliton center-of-mass).

It is important to note that the instability of the translational mode does not mean instability of the soliton structure.

We have to say that the frequency of the oscillations observed in the numerical simulations coincides with the one obtained theoretically using Eq. (6). The frequency of the width oscillations can be obtained using the equations $\omega_1 = \sqrt{\Gamma_1}$, where $\Gamma_1 = B^2(3A - 1) - 1$.

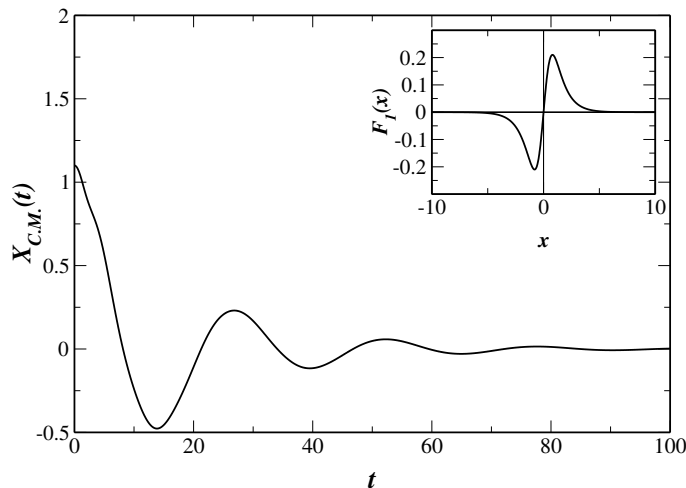


Fig. 1. Soliton center of mass dynamics for the stable case ($B^2 > 1$). The inset shows $F_1(x)$ for $B^2 > 1$.

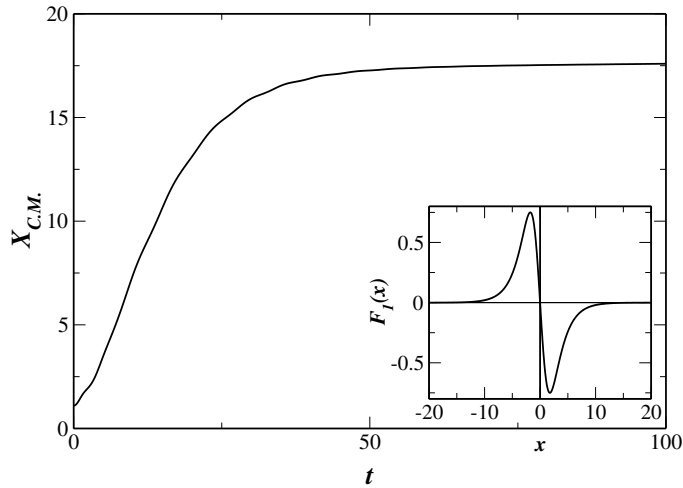


Fig. 2. Soliton center of mass dynamics for the unstable case ($B^2 < 1$). The inset shows $F_1(x)$ for $B^2 < 1$.

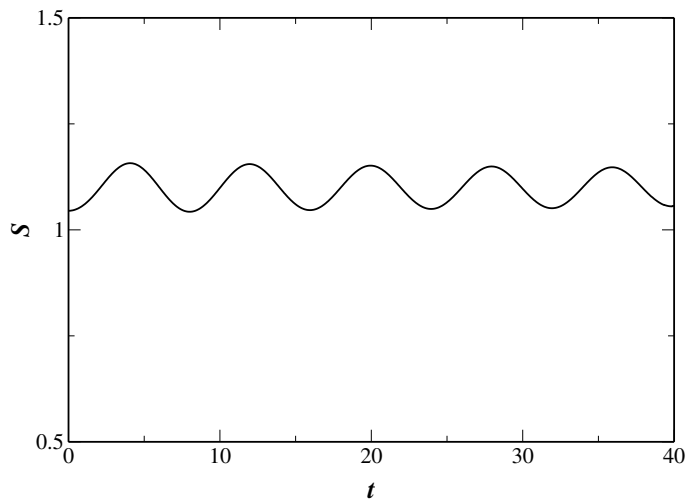


Fig. 3. Soliton's width oscillations when the internal mode can be excited and it is stable, $1/6 < B^2 < 1/3$.

The most spectacular phenomenon occurs for $B^2 < 2/A_*(A_* + 1)$, $A_* = (5 + \sqrt{17})/2$. In this case, the first internal mode is unstable. If we study the evolution of the soliton from the initial conditions (9) and (10) we will observe the destruction of the soliton (see Fig. 4). Two solitons move away (in different directions) to “infinity” (or to the boundaries of the system) and an antisoliton is formed in the place of the original soliton remaining there stabilized. In fact, the condition $B^2 < 1$ implies stability for the center-of-mass of an antisoliton.

Note that in this situations, the sine-Gordon solitons do not behave as rigid objects, which is what is expected from them in general [33].

4. Double-well potential for the soliton

The initial distortions of the width of the soliton will eventually be damped due to dissipation.

Once the internal modes are possible, as in Eqs. (2) and (3) with $1/6 < B^2 < 1/3$, we need time-dependent external forces to sustain the oscillations of the soliton width.

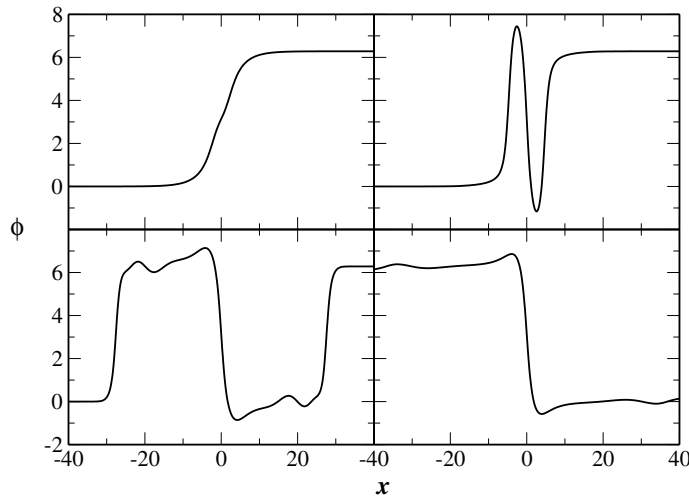


Fig. 4. Soliton’s destruction when the internal mode is unstable, $B^2 < 2/\Lambda_*(\Lambda_* + 1)$, where $\Lambda_* = (5 + \sqrt{17})/2$.

On the other hand, if we wish the soliton to remain in some spatially localized zone, we need stable equilibrium positions for the center-of-mass of the soliton.

Let us consider the following spatiotemporal perturbation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin \phi = F_2(x) + F_3(x, t), \tag{11}$$

where

$$F_2(x) = \begin{cases} F_1(x), & \text{if } -x_1 \leq x \leq x_1, \\ \frac{A}{\cosh[B(x+x_1)]} - D, & \text{if } x < -x_1, \\ D - \frac{A}{\cosh[B(x-x_1)]}, & \text{if } x > x_1 \end{cases} \tag{12}$$

and $F_3(x, t) = f_0 \cos(\omega t)g(x)$, where $g(x) = 1/\cosh^2(E(x + x_1)) + 1/\cosh^2(E(x - x_1))$.

The space-dependent force $F_2(x)$ creates a double-well potential for the soliton.

At the same time, in the interval $-x_1 < x < x_1$, we have the same force $F_1(x)$, which was sufficient for the existence of the internal mode.

Actually, other forces can be used to excite the internal mode. In fact, if we have a function $F(x)$ that can mimic approximately the behavior of function $F_1(x)$ (specially in the interval $-x_1 < x < x_1$, where $-x_1$ and x_1 are the extrema of function $F_1(x)$) when B satisfies the condition $1/6 < B^2 < 1/3$, then this function is good for exciting the internal mode. And note that the behavior of function $F_1(x)$ in the interval $-x_1 < x < x_1$ is a very common behavior for a function in an interval where there is a zero and two extrema.

If in this double-well potential the middle (unstable) equilibrium position is such that the condition for the instability of the first internal mode holds, then we can observe again the destruction of the soliton (see Fig. 5).

As before, an antisoliton is created in the position $x = 0$. However, in this case the “new” two solitons do not move away from the inhomogeneities. They remain trapped in the wells.

The time-dependent force $F_3(x, t)$ will cause the soliton width to oscillate. The center-of-mass of the soliton will also oscillate.

In some cases, the soliton center of mass can oscillate, but these oscillations will remain confined inside one of the potential wells (see Fig. 6). On the other hand, if we slightly increase the amplitude of the driving force, we can make the soliton center of mass to jump between the potential wells created by force $F_2(x)$ (see Fig. 7).

Although the soliton is not always in the interval $-x_1 < x < x_1$, it will return to this interval regularly. While the soliton is in this interval, all the conditions hold for the internal mode to be excited.

A very small (further) increase in the amplitude of the driving force will cause extraordinary deformations of the soliton shape (see Fig. 8). Note that in some instances, the soliton can be destroyed, but in this case, this phenomenon is not permanent, because the soliton can be later reconstructed. This is a very dynamical structure with alternating destructions and reconstructions. The center of mass of the structure will be jumping between the potential wells.

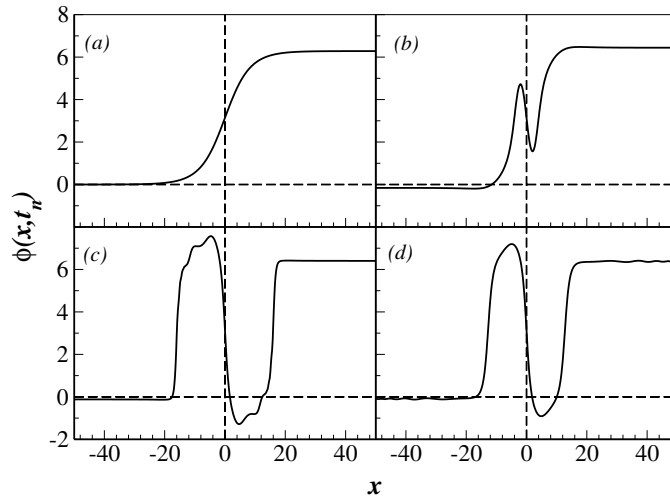


Fig. 5. Soliton destruction inside the double-well potential (Eq. (11) with $B = 0.2, D = 0.0, \gamma = 0.1, C = -0.1, f_0 = 0.0, E = 0.2$).

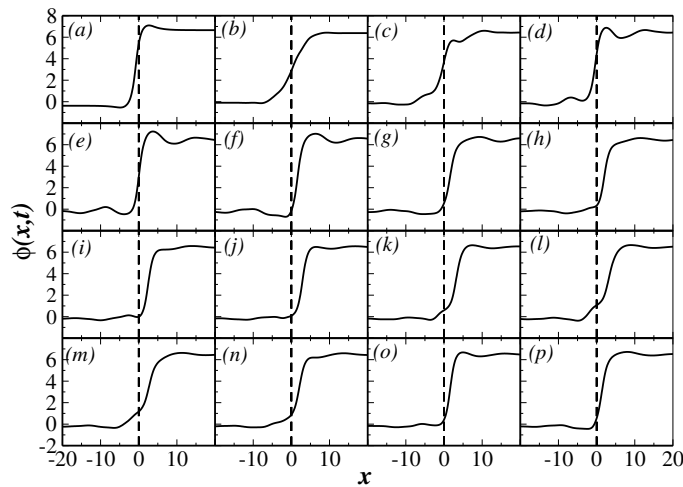


Fig. 6. Soliton shape dynamics inside the right potential well (Eq. (11) with $B = 0.5, D = 0.2, x_0 = 0.0, v_0 = 0.0, \gamma = 0.1, C = 0.0, f_0 = 0.55, \omega = 0.55, E = 0.99$). The snapshots correspond to different time instants.

In all these situations the initial soliton is not deformed. All the oscillations of the soliton width are produced by the time-dependent external forces.

We will now change the spatio-temporal force used to drive Eq. (11) with the following:

$$F_3(x, t) = f_0 \cos(\omega t) \sinh(Bx)g(x), \tag{13}$$

where $g(x) = 1/\cosh^2(B(x + x_1)) + 1/\cosh^2(B(x - x_1))$. If f_0 is too large (see Fig. 9) the soliton will be immediately destroyed and the “three-particle” structure will be a permanent state. There will be oscillations in this structure but it always will be conformed by an antisoliton and two solitons.

When f_0 is not too large, we can have a situation similar to that of Fig. 8 (see Fig. 10) where there are large deformations of the soliton profiles with sporadic destructions of the soliton and eventual reconstructions.

The spatiotemporal force $F_3(x, t)$ in Eq. (11) is chosen such that it acts directly on the internal mode. However, we have corroborated that other spatiotemporal forces also can sustain the soliton width oscillations. This includes the “ubiquitous” force $F_3(t) = f_0 \cos(\omega t)$.

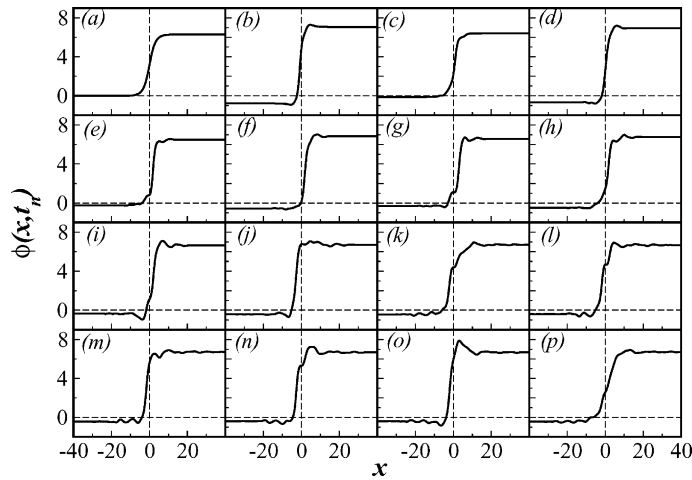


Fig. 7. Soliton shape dynamics when the center of mass is jumping between the two wells (Eq. (11) with $B = 0.5$, $D = 0.4$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = 0.0$, $f_0 = 0.6$, $\omega = 1.0$, $E = 0.8$).

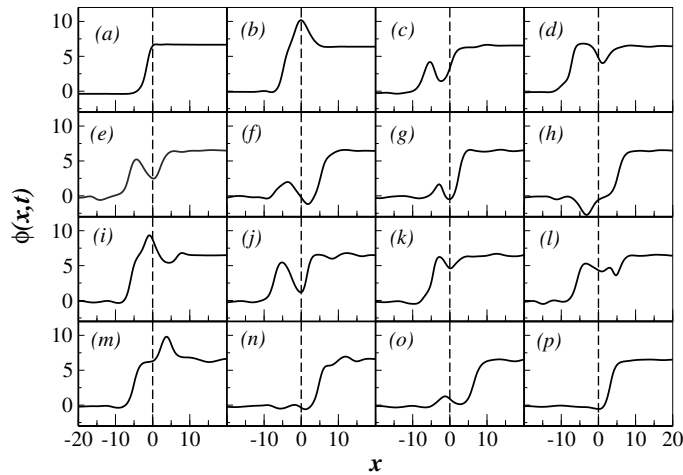


Fig. 8. Soliton profiles for different time instants governed by Eq. (11) ($B = 0.5$, $D = 0.2$, $\gamma = 0.1$, $f_0 = 0.7$, $\omega = 0.55$, $E = 0.7$). Note the large deformations in soliton shape.

Here we should say that using the force (13) we can produce harmonic oscillations of the soliton width (see Fig. 11) even when the center of mass of the soliton is oscillating inside a stable potential well. This result was obtained using the following equation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F_1(x) + F_3(x, t), \tag{14}$$

where $F_1(x) = B \sinh(Bx) / \cosh^2(Bx)$, $F_3(x, t) = f_0 \cos(\omega t) \sinh(Bx) / \cosh^2(Bx)$. Note that in general we will need in all cases spatiotemporal perturbations in order to produce oscillations of the shape modes. In particular, in this case where the equilibrium position is stable, a spatiotemporal force that acts directly on the first internal mode is required.

Now we will return to the system where the soliton is moving in a double-well potential created by an external space-dependent force F_2 (12):

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F_2(x) + F_3(x, t), \tag{15}$$

where $F_3(x, t) = f_0 \cos(\omega t)g(x)$, where $g(x) = 1 / \cosh^2(E(x + x_1)) + 1 / \cosh^2(E(x - x_1))$. In addition, $B = 0.5$, $D = 0.4$, $\gamma = 0.1$, $f_0 = 0.6$, $\omega = 1$, and $E = 0.8$. Here the parameters are chosen in such a way that the soliton center of mass will have a dynamics similar to a Duffing equation [34]:

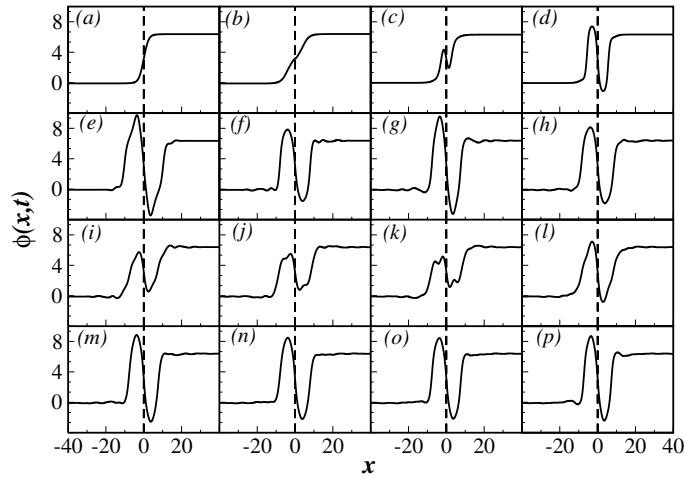


Fig. 9. Permanent soliton destruction due to the action of the spatiotemporal force (13) which is associated to the first shape mode ($B = 0.5, D = 0.1, x_0 = 0.0, v_0 = 0.0, \gamma = 0.1, C = -0.1, f_0 = 0.4, \omega = 1.0$).

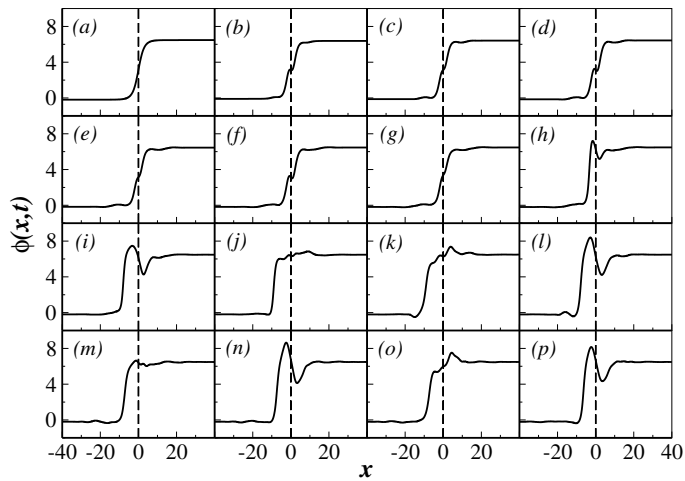


Fig. 10. The same as in Fig. 9 but with a smaller amplitude of the driving force ($B = 0.5, D = 0.1, x_0 = 0.0, v_0 = 0.0, \gamma = 0.1, C = -0.1, f_0 = 0.2, \omega = 1.0$).

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} - x + x^3 = f_0 \cos(\omega t). \tag{16}$$

In this case, the soliton center of mass will perform chaotic oscillations jumping between the two wells created by force F_2 (see Figs. 12 and 13). However, the most interesting result here (considering that our main subject is the internal mode) is the fact that we can observe chaotic oscillations of the soliton width (see Fig. 14).

5. Disordered systems

The propagation of solitons in disordered media has been studied intensively in recent years [29,35]. Consider the equation

$$\phi_{tt} + \gamma \phi_t - \phi_{xx} + \sin \phi = F(x), \tag{17}$$

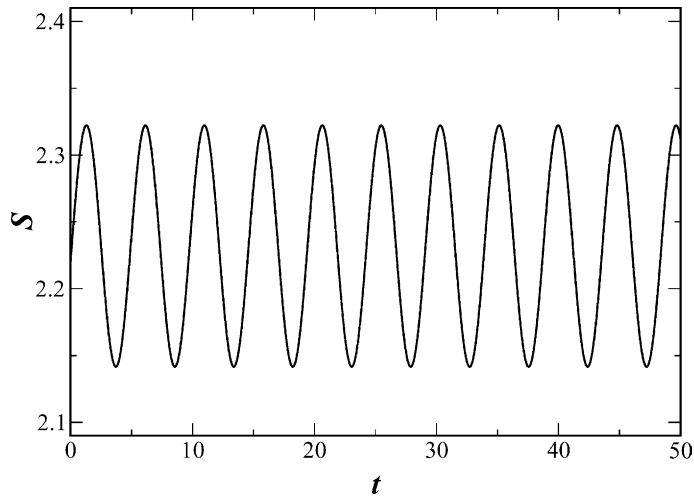


Fig. 11. Time series of the soliton width for a one well potential, see Eq. (14) ($B = 0.5$, $\alpha = 0.0$, $f_0 = 0.1$, $\omega = 1.0$).

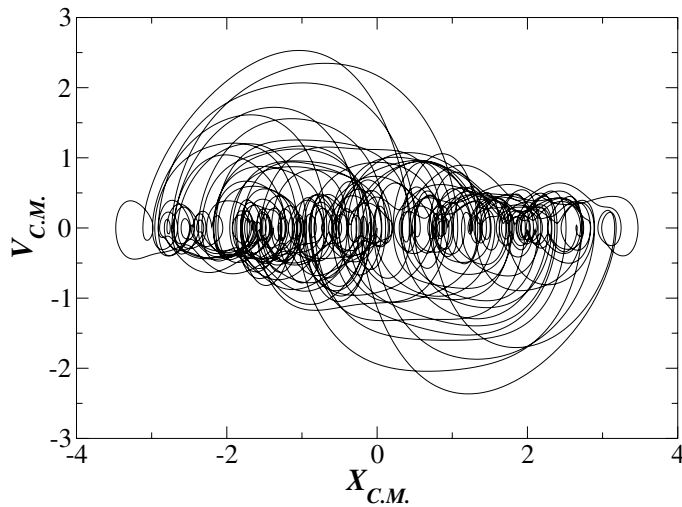


Fig. 12. Phase portrait produced using the dynamics of the soliton center of mass governed by Eq. (15) ($B = 0.5$, $D = 0.4$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = 0.0$, $f_0 = 0.6$, $\omega = 1.0$, $E = 0.8$).

where $F(x)$ is defined in such a way that it possesses many zeroes, maxima and minima (see Fig. 15). This system describes an array of inhomogeneities.

For our study, we have defined $F(x)$ in the following way:

$$F(x) = \sum_{n=-q}^q 4(1 - B^2) \frac{e^{B(x+x_n)} - e^{3B(x+x_n)}}{(e^{2B(x+x_n)} + 1)^2}, \tag{18}$$

where $x_n = (n + 2) \log(\sqrt{2} + 1)/B$ ($n = -q, -q + 1, \dots, q - 1, q$), and $q + 2$ is the number of extrema points of $F(x)$.

In our array, there is a “superposition” of the “disorder” with a dc component which will cause the soliton to move to the right all the time.

When the soliton is moving over intervals where $dF(x)/dx < 0$, the internal mode can be excited. In fact, the points x_i where $F(x_i) = 0$ and $dF(x_i)/dx < 0$, are “barriers” which the soliton can overcome due to its kinetic energy. These “collisions” with the barriers will excite the internal modes if in these intervals the function $F(x)$ mimics the behavior of $F_1(x)$ when $1/6 < B^2 < 1/3$.

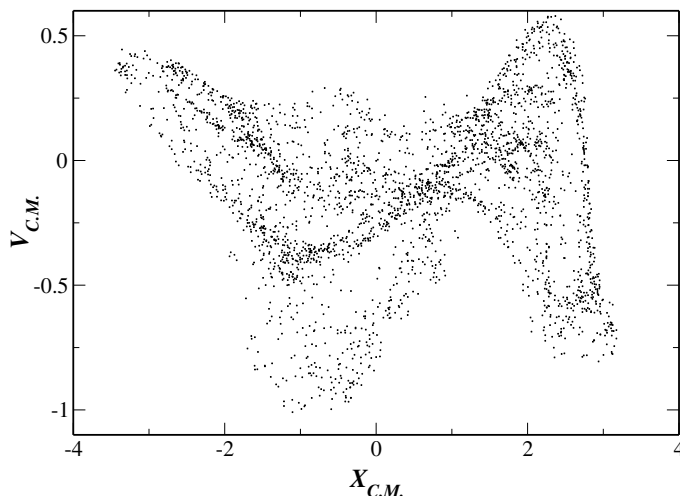


Fig. 13. Poincaré map for the dynamics of Fig. 12 ($B = 0.5$, $D = 0.4$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = 0.0$, $f_0 = 0.6$, $\omega = 1.0$, $E = 0.8$).

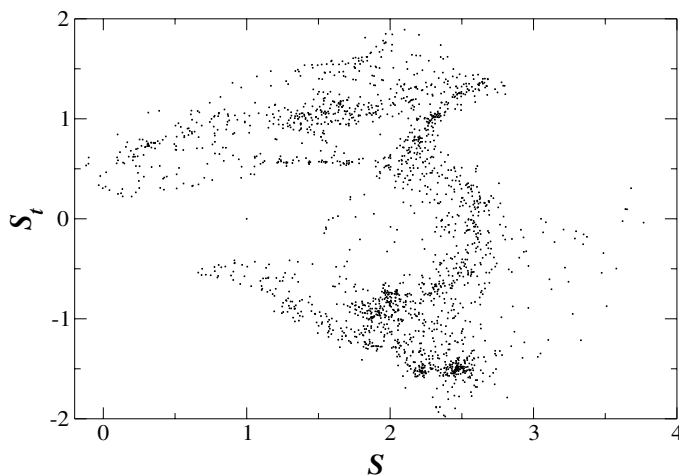


Fig. 14. Poincaré map produced using the dynamics of the soliton width (see Eq. (15)) ($B = 0.5$, $D = 0.4$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = 0.0$, $f_0 = 0.6$, $\omega = 1.0$, $E = 0.8$).

In Fig. 16 we can see how the width of the soliton will perform sustained oscillations during its motion in a disordered medium.

6. Discussion and conclusions

We have shown that in sine-Gordon equations perturbed by inhomogeneous (space-dependent) forces $F(x)$, the solitons can possess internal modes.

Our results are in agreement with previous works [22,33]. In fact, in Eq. (2), with $F_1(x)$ as in (3), if we put $B^2 = 1$, then there are no external forces, and from Eq. (6) we obtain that there are no internal modes in that case neither.

Moreover, having any external inhomogeneous force does not necessarily mean having internal modes.

For instance, if $F(x)$ has a zero which corresponds to a stable equilibrium position for the soliton, even then the internal modes are impossible. This explains why it has been so difficult to find sine-Gordon internal modes.

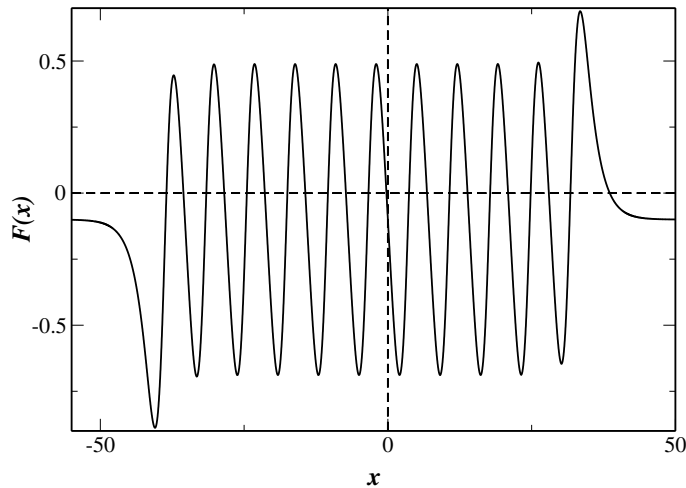


Fig. 15. Array of inhomogeneities (see Eq. (18)).

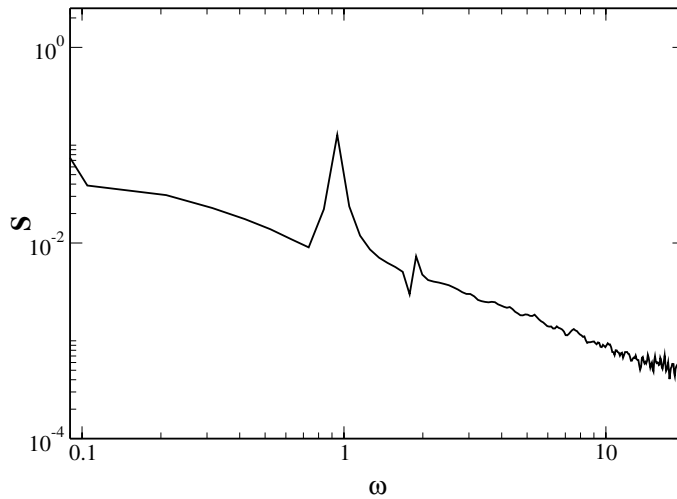


Fig. 16. Fourier transform of the dynamics of the soliton width while it is moving in the array of inhomogeneities (see Eqs. (17) and (18)).

For the existence of internal modes in the sine-Gordon solitons we need zeroes of the function $F(x)$ that corresponds to unstable positions for the center-of-mass of the soliton.

When the soliton center-of-mass is very close to an unstable equilibrium position, there is a pair of forces acting in opposite directions on the “body” of the soliton. This pair of forces should be sufficiently large to stretch the soliton “body”, such that the soliton internal mode can be excited.

Function $F(x)$ can possess several zeroes corresponding to unstable and stable equilibrium positions. For instance, we have studied a force $F(x)$ that creates a double-well potential for the soliton.

Periodic time-dependent forces (along with the spatially inhomogeneous forces) can sustain the oscillations of the soliton width.

A soliton moving in an array of inhomogeneities can also undergo sustained oscillations of its width. All this is possible because the internal mode of the soliton can exist when it is moving in media where there are inhomogeneous space-dependent forces with unstable equilibrium positions.

Nonetheless, we have discovered another more remarkable phenomena. The sine-Gordon internal mode not only can exist for some external forces, but (in some situations) it can become unstable.

If we have an unstable equilibrium position for the soliton center-of-mass and the pair of forces acting on the soliton is too large, then the soliton can be destroyed. When the soliton is destroyed, it can be transformed into an antisoliton and two new solitons. The topological charge is conserved. We had found this phenomenon before for the ϕ^4 -equation [24,25]. However, here for the first time we have shown not only that the sine-Gordon soliton internal mode can exist, but that it can become unstable and destroy the soliton. This is a spectacular manifestation of the fact that the sine-Gordon soliton can behave as a deformable (non-rigid) object.

We should say here that an effective potential for the soliton with stable and unstable equilibrium positions can be created also with the following perturbation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = H(x)\sin\phi, \quad (19)$$

where the extrema of function $H(x)$ can correspond approximately to the equilibrium positions for the soliton (i.e. not the zeroes as in the case of force $F(x)$). With some appropriate functions $H(x)$, the internal mode of the soliton can be excited too.

In Ref. [3], and the references cited therein, we can find many physical applications for such a system.

It has been recognized that the internal mode is a fundamental concept for the non-integrable soliton models [11]. The internal modes of solitons govern the resonant soliton collisions and the soliton-impurity interactions [7–10].

The question about the conditions for creating the soliton's internal mode has interested physicists very strongly [11–14,22]. Several authors [11–14] have shown how the internal mode can appear due to either a small deformation of non-linearity or a weak discreteness. However, this is the first time that external perturbations are shown to be able to create an internal mode for the sine-Gordon soliton.

The existence of the internal mode is very relevant to the soliton dynamics because it can temporarily store part (or all) of the soliton's kinetic energy, which can later be re-stored again in the translational mode. This is important in the soliton–antisoliton collisions [7–9].

The coupling between the translational and the internal modes is the governing factor in many contexts, such as the interaction with inhomogeneities, with thermal noise, and with external drivings.

The soliton internal shape modes can contribute to the thermodynamic properties of the collective soliton–phonon gas [36,37].

Considering all these facts, we believe that the proof of the existence of sine-Gordon internal modes in the presence of spatiotemporal perturbations is a fundamental result.

References

- [1] Lonngren K, Scott A, editors. *Solitons in Action*, Academic, London, 1978.
- [2] Bishop AR, Krumhansl JA, Trullinger SE. *Physica D* 1980;1:1.
- [3] Kivshar YS, Malomed BA. *Rev Mod Phys* 1989;61:763.
- [4] Scott AC. *Nonlinear Science*. Oxford: Oxford University Press; 1999.
- [5] McLaughlin DW, Scott AC. *Phys Rev A* 1978;18:1652.
- [6] Braun OM, Kivshar YS. *Phys Rep* 1998;306:1.
- [7] Peyrard M, Campbell DK. *Physica D* 1983;9:33.
- [8] Campbell DK, Schonfeld JF, Wingate CA. *Physica D* 1983;9:1.
- [9] Campbell DK, Peyrard M, Sodano P. *Physica D* 1986;19:165.
- [10] Kivshar YS, Fei Z, Vázquez L. *Phys Rev Lett* 1991;67:1177.
- [11] Kivshar YS, Pelinovsky DE, Cretegnny T, Peyrard M. *Phys Rev Lett* 1998;80:5032.
- [12] Braun OM, Kivshar YS, Peyrard M. *Phys Rev E* 1997;56:6050.
- [13] Kevrekidis PG, Jones CKRT. *Phys Rev E* 2000;61:3114.
- [14] Kapitula T, Kevrekidis PG, Jones CKRT. *Phys Rev E* 2001;63:036602.
- [15] Rice MJ. *Phys Rev B* 1983;28:3587.
- [16] Boesch R, Willis CR. *Phys Rev B* 1990;42:2290.
- [17] Tchofo Dinda P, Boesch R, Coquet E, Willis CR. *Phys Rev B* 1992;46:3311.
- [18] Fei Z, Konotop VV, Peyrard M, Vázquez L. *Phys Rev E* 1993;48:548.
- [19] Zhang F. *Phys Rev E* 1996;54:4325.
- [20] Zhang F. *Physica D* 1997;110:51.
- [21] Majerníková E, Gaididei YB, Braun OM. *Phys Rev E* 1995;52:1241.
- [22] Quintero NR, Sánchez A, Mertens FG. *Phys Rev E* 2000;62:R60.
- [23] González JA, Holyst JA. *Phys Rev B* 1992;45:10338.
- [24] González JA, Mello BA. *Phys Lett A* 1996;219:226.
- [25] Guerrero LE, Bellorín A, Carbó JR, González JA. *Chaos, Solitons & Fractals* 1999;10:1491.

- [26] González JA. *Mod Phys Lett B* 1992;6:1867.
- [27] González JA, Guerrero LE, Bellorín A. *Phys Rev E* 1996;54:1265.
- [28] González JA, Mello BA, Reyes LI, Guerrero LE. *Phys Rev Lett* 1998;80:1361.
- [29] González JA, Bellorín A, Guerrero LE. *Phys Rev E* 1999;60:R37.
- [30] Aslamasov LG, Gurovich EV. *JETP Lett* 1984;40:746.
- [31] Brazovsky SA, Bak P. *Phys Rev B* 1978;17:3154.
- [32] Flügge S. *Practical Quantum Mechanics*. Berlin: Springer-Verlag; 1971.
- [33] Quintero NR, Sánchez A, Mertens FG. *Phys Rev E* 2001;64:046601.
- [34] Guckenheimer J, Holmes P. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer-Verlag; 1990.
- [35] Gredeskul SA, Kivshar YS. *Phys Rep* 1992;216:1.
- [36] Currie JF, Krumhansl JA, Bishop AR, Trullinger SE. *Phys Rev B* 1980;22:477.
- [37] Giachetti R, Sodano P, Sorace E, Tognetti V. *Phys Rev B* 1984;30:4014.