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Internal degrees of freedom, long-range interactions and nonlocal effects in perturbed Klein–Gordon equations

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ABSTRACT

We investigate different mechanisms for the excitation of soliton internal degrees of freedom and for the existence of long-range interactions between solitons. We will study a nonlocal Klein–Gordon equation that is used as a model for Josephson junctions in thin films. We will show the connections between nonlocality, nonlinearity, internal degrees of freedom, long-range interactions and power-law behaviors.

1. Introduction

Recently, a wealth of work has been dedicated to solitons and solitary waves [1–4]. A much discussed question is the existence of internal modes of solitary waves [5–13]. Internal modes introduce fundamentally new phenomena into the system dynamics as, for example, long-lived oscillations of the solitary wave shape and resonant solitary wave interactions [5–13].

A very important question is: what are the mechanisms for the creation of solitary wave internal modes?

Some attention has been devoted to the following mechanisms: discreteness, deformation of the potential that corresponds to the nonlinearity and higher-order dispersion [9].

Herein we will address new mechanisms and their interaction with other phenomena and concepts: space-dependent external inhomogeneous perturbations, nonlocal operators, and long-range interactions.

We will investigate novel physical effects that are possible when new internal modes of solitary waves are created, and we will show that some of the most spectacular novel phenomena are produced by the internal mode instabilities.

The existence of internal modes is very relevant also to the question related to intrinsic localized modes and discrete breathers [9,14–16]. This is a very popular theme in current scientific literature.

Some of the results presented here are related to more general problems: long-range interactions of solitary waves (many space scales), interaction of nonlinear waves with irregular (fractal) structures, and soliton propagation in disordered media.

We will consider propagation of solitary waves in media where the width of the solitary wave is comparable with the characteristic length of the media. Recall the analogy with the applications of geometrical optics—geometrical optics can be applied only when the wave length is much less than the length of any change in the media.

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Our paper is organized as follows. In Section 2 we present the model equations. The dynamics of kinks in inhomogeneous media is described in Section 3. In Section 4, we analyze connections between the number of internal modes and the character of the kink–antikink interactions. A nonlocal Klein–Gordon equation is studied in Section 5. The main results of the paper are discussed in these three Sections 3–5. This information can be summarized in the following sentence: The kink width and the asymptotic behavior of the kink are related to the number of internal modes. In Section 6, an analysis of the competition of mechanisms is presented. Finally, in Section 7 we summarize and present our conclusions.

2. The model equations

As a model equation, we will investigate a nonlinear Klein–Gordon equation:

$$\phi_{tt} + \gamma\phi_t - \widehat{N}\phi + \frac{\partial U(\phi)}{\partial\phi} = F(x,t), \tag{1}$$

where $U(\phi)$ is a potential that possesses at least two minima in points ϕ_1 and ϕ_3, and a maximum in point ϕ_2, such that $\phi_1 < \phi_2 < \phi_3$, and $U(\phi_1) = U(\phi_3)$. This is a needed condition for the existence of kink/anti-kink solutions.

In its simplest case, operator $\widehat{N}$ is the second space derivative: $\widehat{N} = \partial_{xx}$. In the present paper we will also consider situations where $\widehat{N}$ is a nonlocal operator, for example:

$$\widehat{N}\phi = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} Q(x,s)\frac{\partial\phi}{\partial s}\mathrm{d}s. \tag{2}$$

The sine–Gordon and ϕ^4 equations are particular examples of Eq. (1).

It is well-known that most studies of the long Josephson junction are based on the local sine–Gordon equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = 0. \tag{3}$$

A mechanical analog of this model is a chain of coupled pendula in the presence of damping and forcing, in which the term ϕ_{xx} represents the interaction between pendula.

In situations where the electromagnetic fields penetrate the superconducting electrodes and extends into free space, and the fields outside the junction contribute significantly to the junction energy, then the local sine–Gordon equation is no longer valid. In these situations we should use the following nonlocal sine–Gordon equation for describing the junction:

$$\phi_{tt} + \gamma\phi_t - \lambda_j^2\frac{\partial}{\partial x}\int Q(x,s)\frac{\partial\phi}{\partial s}\mathrm{d}s + \sin\phi = F(x,t). \tag{4}$$

There are very important investigations [17] of Josephson junctions in thin films where this equation is presented as the correct model (see also the references [18,19] where the nonlocal electrodynamics in Josephson junctions is studied).

The kernel $Q(x,s)$ depends on the experimental situations (see [18,19]). Suppose that $Q(x,s) = Q(y)$ (where $y = |x-s|$) is a bell-shaped function. Then the "width" of function $Q(y)$ is a measure of the system nonlocality. Note that for $Q(y) = \delta(y)$, we recover the local equation with $\widehat{N} = \partial_{xx}$.

3. Inhomogeneous systems

In this section, we will investigate the behavior of kinks in the presence of inhomogeneous forces. Consider the following equation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \frac{\partial U(\phi)}{\partial\phi} = F(x). \tag{5}$$

The zeros of $F(x)$ are candidates for equilibrium points for the kink motion.

Suppose $x = x^*$ is a zero of $F(x)$ such that $F(x^*) = 0$. If $(\partial F/\partial x)_{x=x^*} > 0$, then $x = x^*$ is a stable equilibrium position. Otherwise, the equilibrium position is unstable (the opposite happens for the antikink).

The number of internal modes can be affected by the presence of inhomogeneous external perturbations.

If the center of mass of the kink is near a stable equilibrium position, the kink is inside a potential well. When the forces that act on the kink from both sides are large enough, the kink can be so tightly compressed that any motion is difficult. Thus the number of internal modes can be reduced when the kink is near a stable equilibrium position.

Meanwhile, when the kink is near an unstable equilibrium position, the kink can be stretched by a pair of forces acting on opposite directions and the number of internal modes can be increased. We will illustrate these phenomena using some exactly solvable models.

We will discuss here the following model for which the problem about the existence of internal modes can be solved exactly.

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - \frac{1}{2}\left(\phi - \phi^3\right) = F(x), \tag{6}$$

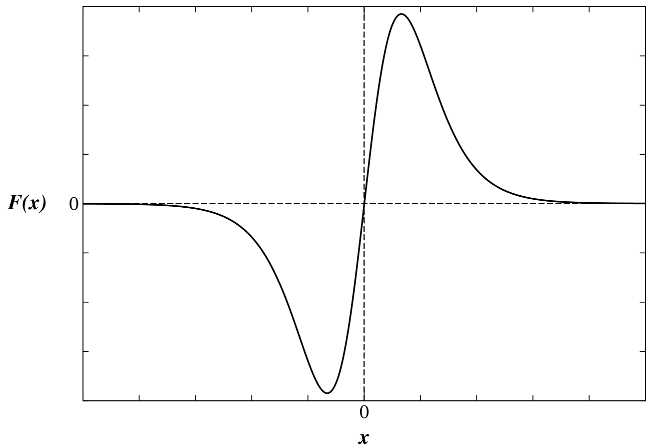

Fig. 1. Perturbation $F(x)$ given by Eq. (8) when $4B^2 > 1$.

where we have chosen $U(\phi)$ and $F(x)$ as follow:

$$U(\phi) = \frac{1}{8}(\phi^2 - 1)^2, \tag{7}$$

$$F(x) = \varepsilon_1 \tanh(Bx) + \varepsilon_2 \sinh(Bx)/\cosh^3(Bx). \tag{8}$$

The force (8) possesses the convenient property that for different values of the parameters function $F(x)$ can have one or three zeros. These zeros can be equilibrium positions for the center of mass of the kink.

In order to simplify the calculations we will put $\varepsilon_1 = \frac{1}{2}A(A^2 - 1)$, and $\varepsilon_2 = \frac{1}{2}A(4B^2 - A^2)$.

In this case, the exact solution for a kink equilibrated at position $x = 0$ is very simple: $\phi_k = A\tanh(Bx)$. The kink width is defined as $1/B$.

The question about the existence of solitonic modes [20] leads to the eigenvalue problem

$$\widehat{L}f = \Gamma f, \tag{9}$$

where

$$\widehat{L} = -\partial_{xx} + \left[\frac{3}{2}A^2 - \frac{1}{2} - \left(\frac{3}{2}\right)A^2/\cosh^2(Bx)\right], \tag{10}$$

and

$$\Gamma = -\left(\lambda^2 + \gamma\lambda\right). \tag{11}$$

The spectrum has a discrete and a continuum parts. The eigenvalues that correspond to the discrete spectrum are

$$\Gamma_n = B^2(\Lambda + 2\Lambda n - n^2) - 1/2, \tag{12}$$

where $\Lambda(\Lambda + 1) = 3A^2/2B^2$, $n \le [\Lambda]$ ($[\Lambda]$ is the integer part of Λ). The integer part of Λ gives the number of solitonic modes: this includes the translational mode Γ_0, and the internal shape modes Γ_i ($i \ge 1$). If $A = 1$, then

$$F(x) = \frac{1}{2}(4B^2 - 1)\sinh(Bx)/\cosh^3(Bx) \tag{13}$$

is a simple function with only one zero at point $x = 0$ as shown in Figs. 1 and 2.

For $4B^2 > 1$, the equilibrium position $x = 0$ is stable. Otherwise ($4B^2 < 1$), the equilibrium position is unstable.

As $B \to 0$, the number of solitonic modes increases as the integer part of Λ, where

$$\Lambda = \frac{\sqrt{1 + 6/B^2} - 1}{2}, \tag{14}$$

and the number of internal shape modes is given by $[\Lambda] - 1$.

For example, for $B^2 < 3/25$, the number of internal modes is already two or larger. Recall, for comparison, that the unperturbed ϕ^4 equation has only one internal mode.

In general, the existence of zeros of $F(x)$, x^* such that $(\partial F/\partial x)_{x=x^*} < 0$ can lead to the appearance of many internal shape modes for the kink.

However, inhomogeneous external forces can also decrease the number of solitonic modes.

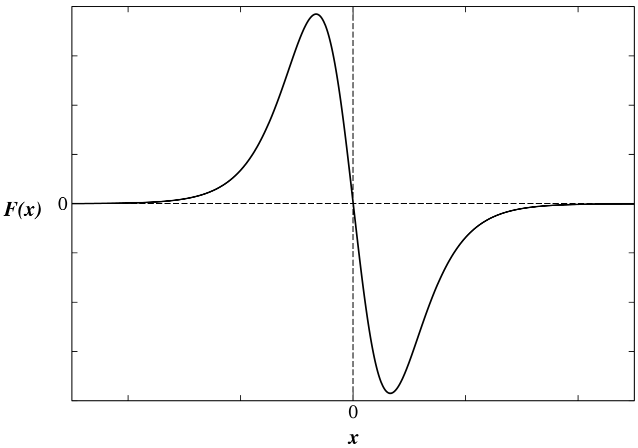

Fig. 2. Perturbation $F(x)$ given by Eq. (8) when $4B^2 < 1$.

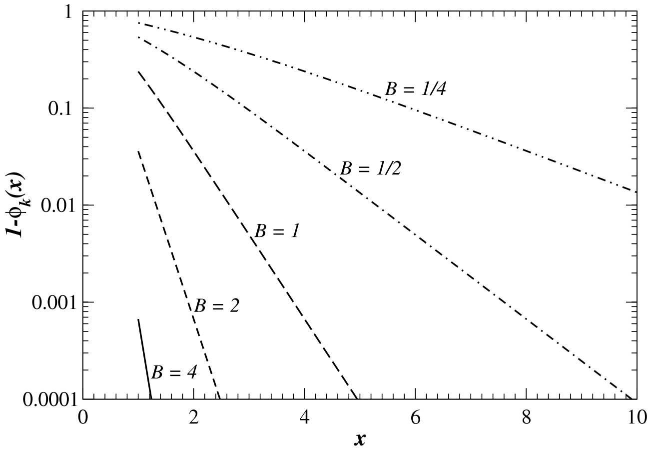

Fig. 3. Asymptotic exponential behavior of the kink solution for different values of B. This is a simple log plot.

Again, function (8) is good for the illustration. But now we will consider the case where $(\partial F/\partial x)_{x=x^*} > 0$. The condition for this is $4B^2 > 1$.

As B is increased, the kink can be very tightly compressed inside the potential well created by $F(x)$. The asymptotic behavior is exponential (see Fig. 3).

For $B^2 > 2/5$ the number of solitonic modes is reduced to zero.

Thus, the number of internal modes can be affected by the presence of external inhomogeneous perturbations. However, how the space-dependent perturbations influence the internal modes is not a trivial question. It depends on the properties of function $F(x)$, and not only the number of internal modes depends on the properties of the external perturbations.

Once there is a zero of $F(x)$ such that $F(x^*) = 0$, and $(\partial F/\partial x)_{x=x^*} < 0$, there exist forces acting on the kink that (under certain conditions) can make the kink unstable leading to a breakup.

Further exploiting our example Eq. (6), for

$$B^2 < \frac{11 - \sqrt{117}}{8}, \tag{15}$$

the first internal mode becomes unstable.

The functions $F(x)$ considered in this section are deterministic. However, many of the obtained results can be very useful for analyzing disordered systems. The concept of stability of the equilibria created by zeros of function $F(x)$ is very important. If the distance between the zeros is sufficiently large, then the stability condition is given by the inequality $(\partial F(x)/\partial x)_{x=x^*} > 0$ as discussed above. If the zeros are very close, the middle zero will have the stability of the two neighbor zeros.

It is sufficient to have zeros of $F(x)$ that correspond to unstable equilibrium positions with the properties discussed in this section in order to have the possibility of the activation of new internal modes.

Moreover, unstable equilibrium positions with certain properties can lead to kink breakup and the creation of kink–antikink–kink threesome.

Consider the following inhomogeneous sine–Gordon equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F(x), \tag{16}$$

where $F(x) = 2(x - x^3)/(1 + x^2)^2$.

This is an inhomogeneous external perturbation that creates an equilibrium position for the kink. However, unlike the perturbations considered in Eq. (13), here $F(x)$ decays algebraically for large $|x|$.

The exact solution for the static kink whose center of mass is equilibrated on the point $x = 0$ is

$$\phi_k(x) = 2\arctan(x) + \pi. \tag{17}$$

The eigenvalue problem discussed in Section 2 in connection with the existence of kink internal modes is now

$$\widehat{L}f = \Gamma f, \tag{18}$$

where

$$\widehat{L} = -\partial_{xx} + V(x), \tag{19}$$

and

$$V(x) = 1 - 2/(1 + x^2). \tag{20}$$

Note that this is "formally" a Schrödinger equation with the potential $V(x)$ as in Eq. (20).

The potential well $V(x) = 1 - 2/(1 + x^2)$ supports an infinite number of bound states [21–24].

Thus, we can state that the kink in the perturbed sine–Gordon Eq. (16) can possess an arbitrarily large number of internal modes!

4. Internal modes and long-range interactions in homogeneous Klein–Gordon equations

The number of excited internal modes is related to the behavior of the solution as $\phi(x, t)$ approaches the vacuum values.

We will consider the following exactly solvable model

$$\phi_{tt} - \phi_{xx} = -\frac{\partial U(\phi)}{\partial \phi}, \tag{21}$$

where

$$U(\phi) = 2\left[\frac{\Delta\phi}{\Lambda B\left(\frac{\Lambda}{2}, \frac{\Lambda}{2}\right)}\right]^2 \left\{I^{inv}\left[\frac{\Lambda}{2}, \frac{\Lambda}{2}, \frac{\phi - \phi_1}{\Delta\phi}\right] I^{inv}\left[\frac{\Lambda}{2}, \frac{\Lambda}{2}, \frac{\phi_3 - \phi}{\Delta\phi}\right]\right\}^{\Lambda} \tag{22}$$

where $B(a, b)$ is the β function, and $I^{inv}(a, b, y)$ is the inverse function of the incomplete β function, $y = I_z(a, b)$, with respect to the argument z. The potential $U(\phi)$ satisfies the requirements for the existence of kink and antikink solutions (see Section 2). Here ϕ_1 and ϕ_3 are the minima of $U(\phi)$ and ϕ_2 is a maximum of $U(\phi)$ such that $\phi_1 < \phi_2 < \phi_3$. On the other hand, $\Delta\phi = \phi_3 - \phi_1$. Λ a is parameter that defines a family of models.

The following kink solution can be obtained

$$\phi_k(x) = \phi_1 + \Delta\phi I\left\{\frac{\Lambda}{2}, \frac{\Lambda}{2}, \frac{1 + \tanh[x/\Lambda]}{2}\right\}, \tag{23}$$

where $I(a, b, z) \equiv I_z(a, b)$ is the incomplete β function. Here the kink width is proportional to Λ.

The stability problem leads to the following eigenvalue problem

$$-f_{xx} + \left(\frac{\partial^2 U}{\partial \phi^2}\right)_{\phi=\phi_k(x)} f(x) = -\lambda^2 f(x). \tag{24}$$

This spectral problem can be solved exactly.

The equation can be rewritten in the form:

$$-\frac{d^2 f(x)}{dx^2} + \left\{1 - \frac{\Lambda + 1}{\Lambda \cosh^2[x/\Lambda]}\right\} f(x) = -\lambda^2 f(x). \tag{25}$$

The number of discrete solitonic modes is given by the integer part of Λ. The number of internal modes is $[\Lambda] - 1$.

As Λ increases, the approaching of the vacuum values by the kink solution is slower.

Thus the way the kink solution approaches the asymptotic values is related to the number of internal modes.

From this we can infer that if the kink solution behaves asymptotically as a power-law, the number of internal modes should be very large.

Let us consider the following equation:

$$\phi_{tt} - \phi_{xx} = -\frac{\partial U(\phi)}{\partial \phi}, \tag{26}$$

where the potential $U(\phi)$ behaves as

$$U(\phi) \sim (\phi - \phi_i)^{2n}, \tag{27}$$

in a vicinity of the minima of the potential, ϕ_i. Here n is a natural number.

We will investigate the long-range interaction between solitons and how this phenomenon is related to the anharmonicity of the potential. In this system, power-law behaviors are possible.

As we have mentioned above (see Section 2), we consider a potential $U(\phi)$ that possesses at least two minima at points ϕ_1 and ϕ_3, such that $U(\phi_1) = U(\phi_3)$.

Suppose that in the vicinity of the minima ϕ_i, the potential behaves as $U(\phi) \sim (\phi - \phi_i)^{2n}$.

For $n = 1$, the asymptotic behavior of the kink solutions is exponential and the interaction force $F_{\text{int}}(d)$ between a kink and an antikink decays exponentially within a distance d between the centers of mass of the solitary waves: $F_{\text{int}} \sim \exp(-Cx)$ (here C is a constant).

The sine–Gordon and ϕ^4 equations belong to this subset of models.

Nevertheless, when $n > 1$, the behavior of the kink solution for $x \to \infty$ is $\phi - \phi_i \sim x^k$, where $k = 1/(1-n)$, $i = 1, 3$. And the interaction force decays as $F_{\text{int}}(d) \sim d^{2n/(1-n)}$. Note that this is a power-law behavior.

In these systems, fractal behavior can be observed at all scales.

Nonlocal interaction in coupled Josephson junction has been studied in Ref. [25]. These models can describe a "world" where there is spontaneous formation of topological objects with long-range interactions, which can create complex structures showing fractal behavior.

Let us discuss noise-induced phenomena in systems where soliton interaction is long-range. We will consider pattern formation in the stochastic equation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \frac{\partial U(\phi)}{\partial \phi} = \eta(x, t), \tag{28}$$

where $\eta(x, t)$ is a white noise with the properties $\langle \eta(x, t)\rangle = 0$, $\langle \eta(x, t)\eta(x', t')\rangle = 2D\delta(t - t')\delta(x - x')$. The noise contains all the scales.

Here the potential $U(\phi)$ satisfies the conditions for the existence of long range interactions discussed above.

This model presents noise-induced pattern formation.

When $n = 1$ and noise is small the soliton–antisoliton pairs are not being created yet and the roughening exponent ξ is zero. After the creation of the solitons we observe a crossover from a non-KPZ behavior ($\xi \sim 0.7 - 0.8$) to a KPZ behavior ($\xi \sim 0.5$) [26]. However, for large scales there is a plateau with $\xi = 0$.

For instance, the sine–Gordon does not eliminate the disorder at large scales.

For $n > 1$, the activated solitons possess long-range interactions.

For $n \gg 1$ self-affinity extends to all scales.

Unlike the case $n = 1$, the surface in case $n = 40$ presents only two self-affine regimes: the anomalous $\xi \sim (0.818)$ and the KPZ-like ($\xi \sim 0.5$). The system displays fractal dynamics at all scales.

Wavelet analysis has shown the existence of coherent structures at all scales [27]. The observed patterns are evidence of fractal order.

Thus, the model (28) experiences a transition to an ordered state associated with the activation of a soliton–antisoliton gas. There is a crossover from an anomalous non-KPZ behavior to a KPZ behavior. However, unlike the sine–Gordon equation, for $n > 1$, the self-affinity extends to all scales.

Note that it is the KPZ-like behavior the one that extends to infinity. This is because the KPZ regime is related to the absence of a mass term in the evolution equation. And this is the case for $n > 1$.

On the other hand, the anomalous regime is due to the existence of soliton solutions. The KPZ equation is not a soliton-bearing system. These considerations explain the existence of two regimes in the Klein–Gordon equation that supports long-range interacting solitons.

We have found that there are other power-law behaviors.

Let us analyze the return of the pendula to the equilibrium position in Eq. (26) due to the motion of a kink. When the potential is $U(\phi) = [(1 - \cos\phi)/2]^2/2$ (see Fig. 4), the kink solution is:

$$\phi = 2\arctan\left[(x - x_0 - vt)/\sqrt{1 - v^2}\right] + \pi. \tag{29}$$

The asymptotic behavior is a power-law (see Fig. 5).

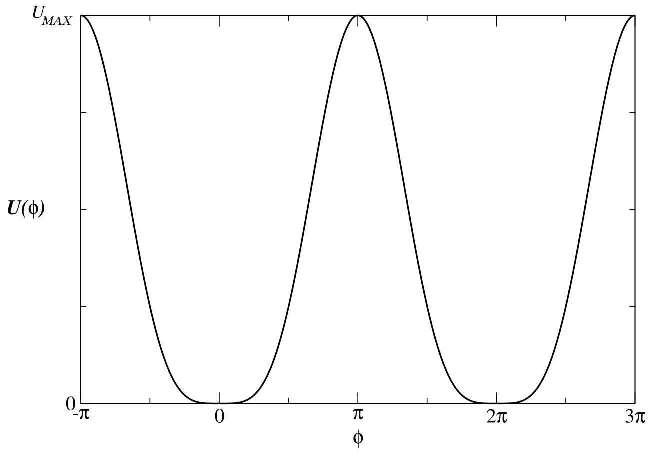

Fig. 4. Potential $U(\phi) = [(1 - \cos\phi)/2]^2/2$ in Eq. (26). The kink–antikink interaction is long range for $n > 1$.

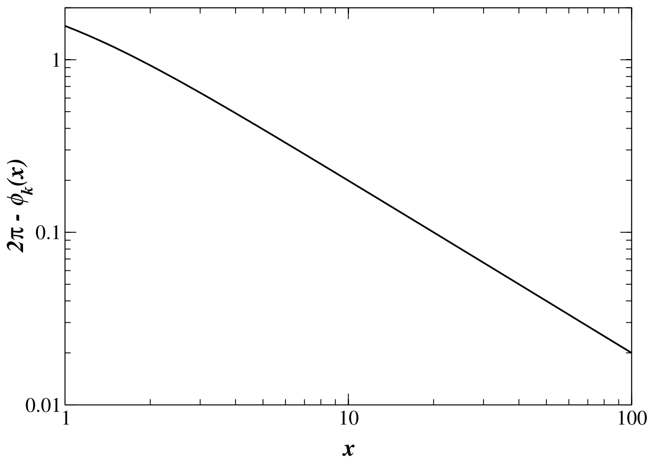

Fig. 5. Asymptotic power law behavior of the kink solution. This is a log–log plot.

Consider the point $\phi(z_0, t)$.

$$\frac{\partial\phi(z_0, t)}{\partial t} \sim -\frac{1}{t^2}. \tag{30}$$

So $\phi(z_0, t) \sim 1/t$.

In this system there is neither a characteristic time, nor a characteristic length, so the processes are relevant at all scales.

Suppose now that we have an overdamped regime:

$$\gamma\phi_t - \phi_{xx} + \frac{\partial U}{\partial\phi} = 0, \tag{31}$$

where $U(\phi) = [(1 - \cos\phi)/2]^2/2$, the relaxation of a single pendulum (without the presence of a kink) will follow the law:

$$\phi(z_0, t) \sim t^{-1/2}. \tag{32}$$

The number of internal modes is proportional to the characteristic interaction length of the soliton.

For the sine–Gordon solitons, the number of internal modes is zero and the equation is completely integrable.

For the ϕ^4 equation, the number of internal modes is equal to one. The equation is considered quasi-integrable.

These solitons can be stretched by external inhomogeneous forces so the effective interaction length and the number of internal modes are increased.

It is possible to establish an heuristic relationship between the nonintegrability of the equation and the number of the internal modes (or the characteristic interaction length).

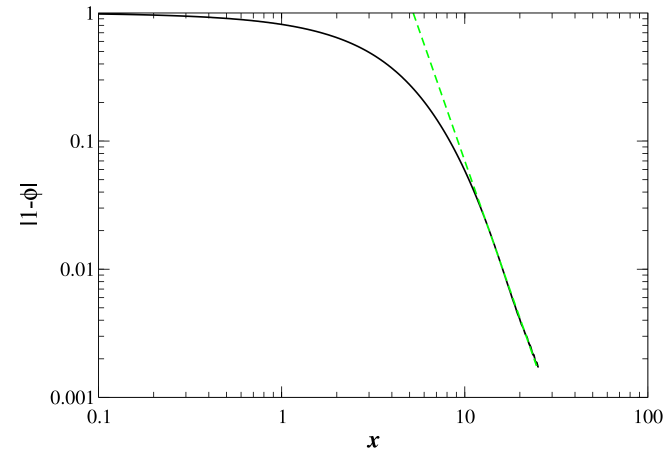

Fig. 6. Log–log plot of the asymptotic behavior of the stabilized solution in the non-local Klein–Gordon equation with $Q(x,y) = a\left[1 + b^2(x-y)^2\right]^{-1}$.

5. Nonlocal Klein–Gordon equation

Recently we have witnessed a growing interest in fractional and nonlocal partial differential equations [28].
In this section we will consider nonlocal equations as the following:

$$\phi_{tt} + \gamma\phi_t - \frac{\partial}{\partial x}\int_{-\infty}^{\infty} Q(x,y)\frac{\partial\phi}{\partial y}\mathrm{d}y = -\frac{\partial U(\phi)}{\partial\phi}. \tag{33}$$

We will show that the solutions possess power-law tails, the character of the kink–antikink interaction is long-ranged.
Define the nonlocal sine–Gordon equation:

$$\phi_{tt} - \frac{\partial}{\partial x}\int_{-\infty}^{\infty} Q(x,y)\frac{\partial\phi}{\partial y}\mathrm{d}y + \sin\phi = 0. \tag{34}$$

The stationary solutions satisfy the equation

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} Q(x,y)\frac{\partial\phi}{\partial y}\mathrm{d}y = \sin\phi, \tag{35}$$

$$Q(x,y) = Q(x-y). \tag{36}$$

Using results from Ref. [29], we arrive at the conclusion that the asymptotic behavior of the solution is governed by the function $Q(x-y)$ at large values of $|x|$.

In the limit $x \to -\infty$, $\phi \approx 2\pi\,\partial Q(x)/\partial x$.

Accordingly, in the limit $x \to \infty$, $\phi \approx 2\pi(1 - \partial Q(x)/\partial x)$.

When $Q(|x-y|)$ decays algebraically, the kink solutions will possess a power-law asymptotic behavior. As we have observed, this phenomenon can lead to long-range interactions and to the generation of internal degrees of freedom. Here we can see possible connections between nonlocality, internal degrees of freedom, power-law behavior and long-range interactions.

In many papers [17,18] on Josephson junctions in thin films, $Q(y)$ decays as $1/y^{\alpha}$ (as shown in Fig. 6).

There is even the kernel $Q(y) \sim \ln y$ [17,18].

In all these cases, $\phi(x)$ possesses a power-law behavior for $x \to \pm\infty$

We can write down an exact solution for the nonlocal sine–Gordon equation when $Q(|x-y|) = (2/\pi)\ln(|x-y|)$: The kink solution is $\phi(x) = 2\arctan(x/2) + \pi$. Compare this function with solutions given by Eqs. (17) and (29).

Now we will analyze the evolution of some initial conditions in the framework of both the local and nonlocal Klein–Gordon Eqs. (33).

Consider the following Cauchy problem: $\phi(x,0) = \tanh(x/2) + A\sin(kx)$ and $\phi_t(x,0) = 0$. This is a deformed kink profile.

As Eq. (33) is a dissipative system, eventually the evolution will lead to the asymptotically stable solution: the basic kink that corresponds to the exact stationary solution.

Fig. 7 shows this process for the local Klein–Gordon equation. The stabilized kink possesses the expected asymptotic exponential behavior for $|x| \to \infty$.

On the other hand, the evolution governed by the nonlocal Eq. (33) with $Q(x,y) = a\left[1 + b^2(x-y)^2\right]^{-1}$ leads to a stabilized kink with power-law asymptotic behavior (see Figs. 6 and 8).

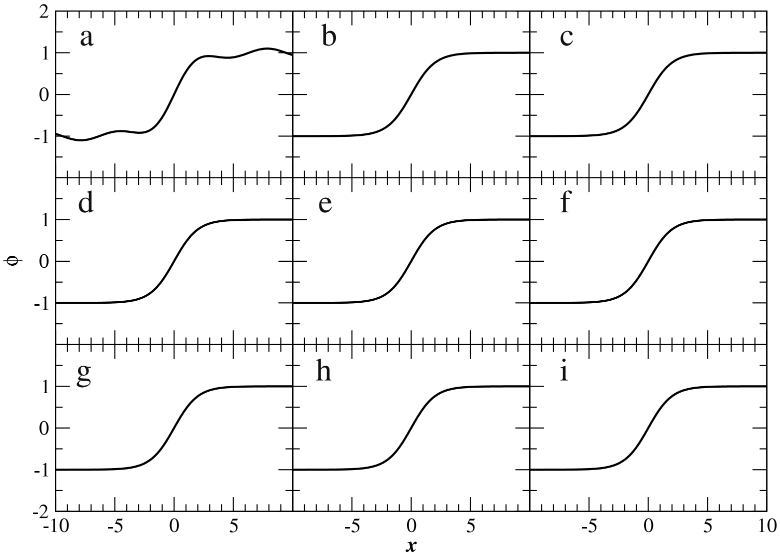

Fig. 7. Evolution of an initially deformed kink profile in the "normal" local Klein–Gordon equation.

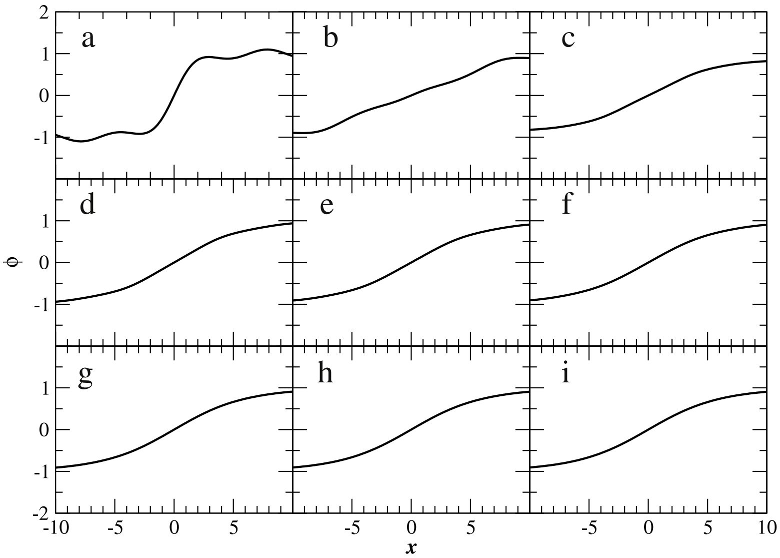

Fig. 8. Evolution of an initially deformed kink profile in non-local Klein–Gordon equation with $Q\,(x, y) = a\left[1 + b^2\,(x - y)^2\right]^{-1}$.

Another interesting Cauchy problem is the following: $\phi(x, 0) = \tanh\left[(x + d)/2\right] - \tanh\left[(x - d)/2\right] + 1$ and $\phi_t(x, 0) = 0$. These initial conditions try to mimic a kink–antikink pair where the kink and the antikink are separated by a distance d.

We will put $\gamma = 0$ in order to better observe the shape variations.

The comparison between the evolution of the kink–antikink pair in the local and nonlocal Klein–Gordon equations (when $d = 2$) is shown in Figs. 9 and 10.

In Fig. 9 (local equation) we can observe the formation of a breather-like structure that slowly decays due to a weak radiation.

Fig. 10 shows that (in the nonlocal case) the interaction is very inelastic, there are many activated internal modes, there exists strong radiation and large waves are created which take away most of the energy to the two extremes of the system.

The dynamics shown in Fig. 11 corresponds to the local equation when the distance between the kink and the antikink is $2d = 10$. Practically, we cannot observe shape changes. The kink–antikink interaction is short-ranged. When d is larger then some critical value, the kink and the antikink do not feel each other. The internal modes are not activated.

In the nonlocal case (Fig. 12), the kink–antikink interaction is long-ranged and many internal modes are activated.

Even when the distance between the kink and the antikink is "large", they will feel the attraction anyway. Eventually, the dynamics will produce the phenomena shown in Fig. 10.

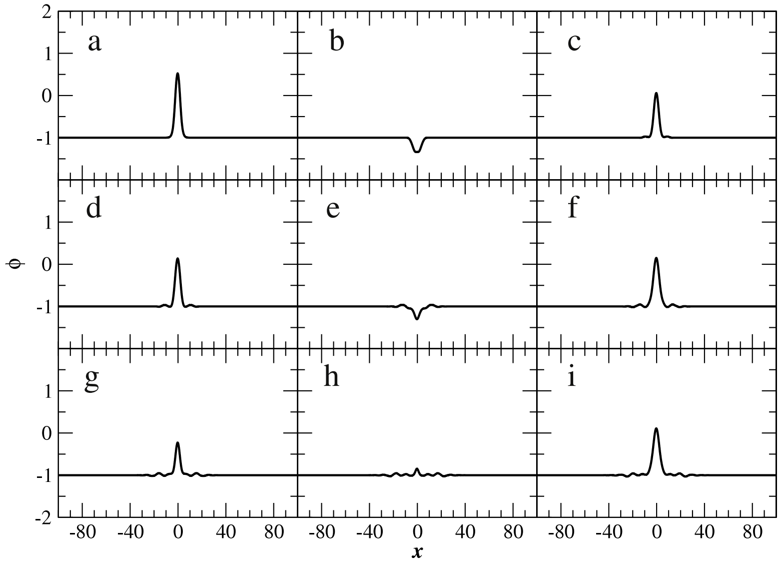

Fig. 9. Evolution of an initially close kink–antikink pair profile in the "normal" local Klein–Gordon equation.

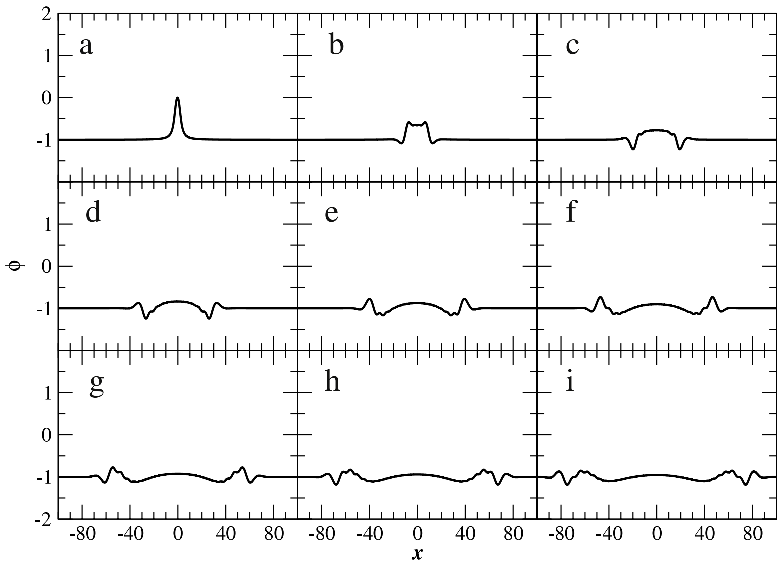

Fig. 10. Evolution of an initially close kink–antikink pair profile in non-local Klein–Gordon equation with $Q\,(x,y) = a\left[1 + b^2\,(x-y)^2\right]^{-1}$.

6. Competition of mechanisms

The effects can be amplified (enhanced) or canceled.

As we have seen, there are different mechanisms that can create kink internal modes. On the other hand, there are mechanisms that can suppress the internal modes.

If the internal mode creating and suppressing mechanisms coexist simultaneously in the system, they can compete.

Mathematically, we can write the following equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \frac{\partial U(\phi)}{\partial \phi} = T_c[\phi, x] + T_s[\phi, x], \qquad (37)$$

where $T_c[\phi, x]$ stands for functional terms that describe internal-mode-creating mechanisms. While $T_s[\phi, x]$ represents functional terms that describe internal-mode-suppressing mechanisms.

In general, we can name several phenomena that generate new internal modes.

(a) Inhomogeneous perturbations containing unstable equilibrium positions for the kink.
(b) Inhomogeneous perturbations that remain "large" for values of x that are far away from the kink center of mass.

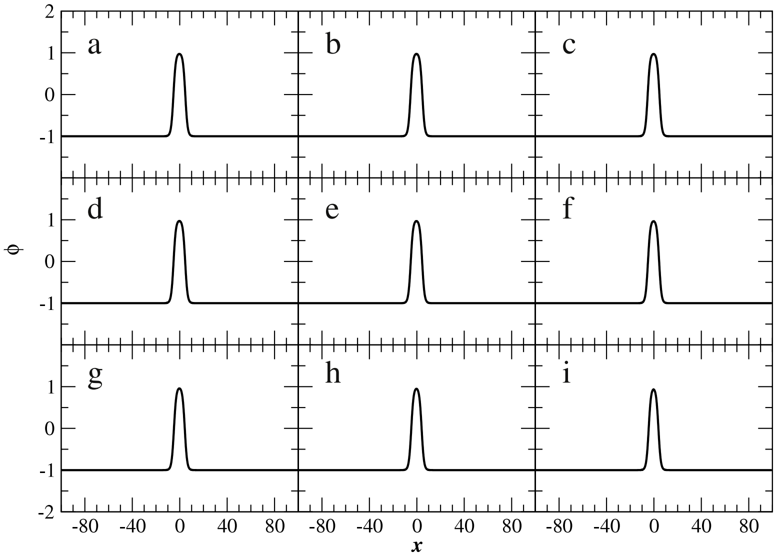

Fig. 11. Evolution of an initially not so close kink–antikink pair profile in the "normal" local Klein–Gordon equation. The kink–antikink interaction is short-ranged. They almost do not "feel" each other. Only after a long time, they start to move feeling the attraction.

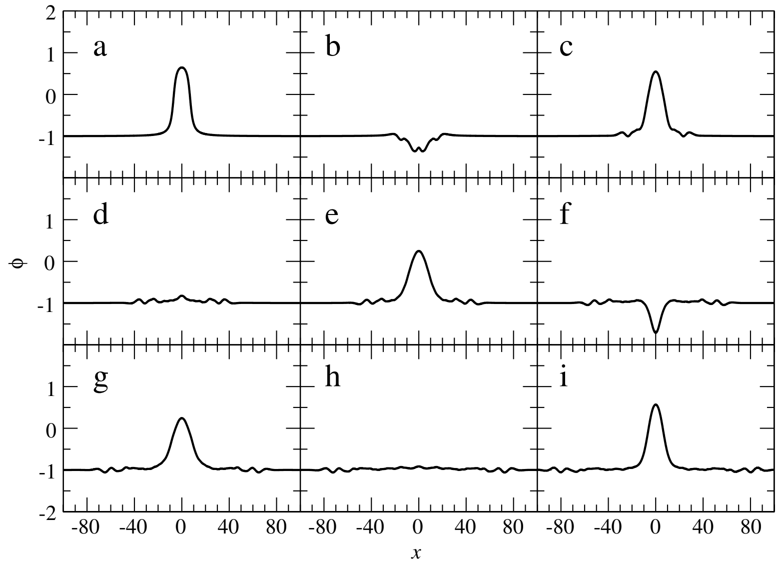

Fig. 12. Evolution of an initially not so close kink–antikink pair profile in non-local Klein–Gordon equation with $Q(x, y) = a\left[1 + b^2 (x - y)^2\right]^{-1}$.

(c) Extended de-localized perturbations
(d) Perturbations that do not decay exponentially
(e) Nonlocal operators.

Note that some of the sets of perturbations can contain or intersect with the other sets.

A simple example of de-localized inhomogeneous perturbation is the following:

$$T_c(x) = \varepsilon_1 \tanh(\beta_1 x). \tag{38}$$

The number of internal modes is proportional to $\varepsilon_1^{1/3}$.

Nevertheless there are perturbations that can reduce the number of internal modes.

For instance, these can be localized perturbations that create a stable equilibrium for the kink.

We will take the following simple example:

$$T_s(x) = \varepsilon_2 \sinh(\beta_2 x)/\cosh^3(\beta_2 x). \tag{39}$$

For large values of β_2, the number of internal modes decays as $1/\beta_2$.

Suppose that both these perturbations are present in Eq. (37).

As in Section 3, in order to simplify the computations, we will redefine the parameters using two new parameters:

$$\varepsilon_1 = \frac{1}{2}A(A^2 - 1), \tag{40}$$

$$\beta_1 = B, \tag{41}$$

$$\varepsilon_2 = \frac{1}{2}A(4B^2 - A^2), \tag{42}$$

$$\beta_2 = B. \tag{43}$$

The calculations yield:

$$\Lambda(\Lambda + 1) = 3A^2/2B^2, \tag{44}$$

where the total number of internal modes is the integer part of Λ.

For a fixed value of B, the larger the value of A, the larger the number of internal modes.

However, even for a fixed large value of A, there is always a critical value B_c such that for $B > B_c$ there are no internal modes at all.

That is, if

$$B^2 > \frac{3}{4}A^2, \tag{45}$$

then

$$\Lambda < 1, \tag{46}$$

which means that there exists no internal modes.

Note that for large values of B, the perturbation $T_s(x)$ given by Eq. (39) is very localized.

7. Conclusions

We have investigated (both theoretically and numerically) different mechanisms that are capable to create new soliton internal modes.

A list of such mechanisms would contain some of the following items: inhomogeneous perturbations that generate unstable equilibrium positions for the kink, extended de-localized space-dependent perturbations, external perturbations that do not decay exponentially, and nonlocal operators.

The link between all our results is the following: *the number of soliton internal modes can depend on the soliton width and the asymptotic behavior of the soliton solution for large values of* $|x|$.

For instance, if the bistable potential $U(\phi)$ (that appears in the generalized Klein–Gordon equation (1)) is deformed in such a way that the asymptotic behavior of the kink solution $\phi_k(x)$ is changed, then this deformation can lead to an increased number of internal modes (see Section 4).

A key issue to be studied is how slowly the function $\phi_k(x)$ approaches the minima of potential $U(\phi)$ for large values of $|x|$.

The kink can be considered as an extended object with a defined width.

If the deformation of the potential produces a "stretching" of the kink (i.e. increased width), then this deformation can generate new internal modes.

This "stretching" of the kink can be produced also by inhomogeneous external space-dependent perturbations $F(x)$ (see Section 3) or by the presence of nonlocal operators (see Section 5).

In fact, if $F(x)$ possesses zeros x^* ($F(x^*) = 0$) such that they represent unstable equilibrium positions for the kink, then new internal modes can be excited. Moreover, under certain conditions, the first internal mode of the kink can be unstable leading to the "destruction" of the soliton.

However, this "destruction" is followed by the generation of a kink–antikink pair and an additional kink in such a way that the topological charge is conserved: the initial kink is replaced by an antikink that remains trapped in the equilibrium position and two kinks that move away.

Nevertheless, we should remark that even when $F(x)$ creates only one stable equilibrium position for the kink, several internal modes can exist if $F(x)$ is not localized. For instance, when $F(x)$ tends to finite values when $x \longrightarrow \pm\infty$.

Furthermore, many internal modes can exist if $F(x)$ behaves as a power-law.

The range of the interaction "force" between a kink and an antikink depends also on the asymptotic behavior of the soliton solutions for large values of $|x|$.

On the other hand, the kink solutions possess a power-law asymptotic behavior when the kernel $Q(x-s)$ in the nonlocal operator that appears in Eq. (1) satisfies some conditions.

All these phenomena lead to interesting connections between nonlocality, internal degrees of freedom, long-range interactions and power-law behaviors.

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