



# Soliton structure dynamics in inhomogeneous media

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## Abstract

We show that soliton interaction with finite-width inhomogeneities can activate a great number of soliton internal modes. We obtain the exact stationary soliton solution in the presence of inhomogeneities and solve exactly the stability problem. We present a Karhunen–Loève analysis of the soliton structure dynamics as a time-dependent force pumps energy into the translational mode of the kink. We show the importance of the internal modes of the soliton as they can generate shape chaos for the soliton as well as cases in which the first shape mode leads the dynamics. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The propagation of solitons in the presence of inhomogeneities concerns a wide variety of condensed matter systems. The traditional approach considers structureless solitons and delta-function-like impurities.

Real scenarios involve finite-width impurities and under certain circumstances, the extended character of the soliton must be considered [1–4]. For instance, the length scale competition between the width of inhomogeneities, the distance between them and the width of the kink-soliton leads to interesting phenomena like soliton explosions [2].

In this paper we take into account the extended character of both the soliton and the impurity and show that these considerations lead to the existence of a finite number

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of soliton internal modes that underlies a rich spatiotemporal dynamics. We present a model for which the exact stationary soliton solution in the presence of inhomogeneities can be obtained and the stability problem can be solved exactly. We use the Karhunen–Loève (KL) decomposition to relate the excitation of soliton internal modes with the sequence of bifurcations obtained as the amplitude of a space-time-dependent driving force (fitted to the shape of the translational mode) is increased.

## 2. The model

The topological solitons studied in the present paper possess important applications in condensed matter physics. For instance, in solid state physics, they describe domain walls in ferromagnets or ferroelectric materials, dislocations in crystals, charge-density waves, interphase boundaries in metal alloys, fluxons in long Josephson junctions and Josephson transmission lines, etc. [5,6].

Although some of the above mentioned systems are described by the  $\phi^4$ -model and others by the sine-Gordon equation (and these equations, in their unperturbed versions, present differences like the fact that the sine-Gordon equation is completely integrable whereas the  $\phi^4$ -model is not) the properties of the solitons supported by sine-Gordon and  $\phi^4$  equations are very similar. In fact, these equations are *topologically equivalent* and very often the result obtained for one of them can be applied to the other [5].

Here we consider the  $\phi^4$  equation in the presence of inhomogeneities and damping:

$$\phi_{xx} - \phi_u - \gamma\phi_t + \frac{1}{2}(\phi - \phi^3) = -N(x)\phi - F(x), \quad (1)$$

where  $F(x)$  is a function with (at least) one zero and  $N(x)$  is a bell-shaped function that rapidly decays to zero for  $x \rightarrow \pm \infty$ . An impurity of the kind  $N(x)\phi$ , but using delta functions, has been presented in Ref. [7].

In ferroelectric materials  $\phi$  is the displacement of the ions from their equilibrium position in the lattice,  $\frac{1}{2}(\phi - \phi^3)$  is the force due to the anharmonic crystalline potential,  $F(x)$  is an applied electric field, and  $N(x)$  describes an impurity in one of the anharmonic oscillators of the lattice [8]. In Josephson junctions,  $\phi$  is the phase difference of the superconducting electrons across the junction,  $F(x)$  is the external current, and  $N(x)$  can describe a microshort or a microresistor [9]. In a Josephson transmission line it is possible to apply nonuniformly distributed current sources ( $F(x)$ ) and to create inhomogeneities of type  $N(x)$  using different electronic circuits in some specific elements of the chain [6,10].

In the present paper the functions  $F(x)$  and  $N(x)$  will be defined as,

$$F(x) = \frac{1}{2}A(A^2 - 1)\tanh(Bx), \quad (2)$$

$$N(x) = \frac{1}{2} \frac{(4B^2 - A^2)}{\cosh^2(Bx)}. \quad (3)$$

The case  $F = \text{const.}$  has been studied in many papers (see e.g. [5]). Here Eq. (2) represents an external field (or a source current in a Josephson junction) that is almost

constant in most part of the chain but changes its sign in  $x=0$  (this is very important in order to have soliton pinning [1]). Microshorts, microresistors or impurities in atomic chains [9] are usually described by Dirac's delta functions ( $\delta(x)$ ) where the width of the impurity is neglected. The function  $N(x)$  is topologically equivalent to a  $\delta(x)$  but it allows us to consider the influence of the width of the impurity.

### 3. Stability analysis

Suppose the existence of a static kink solution  $\phi_k(x)$  corresponding to a soliton placed in a stable equilibrium state created by the inhomogeneities of Eq. (1). We analyze the small amplitude oscillations around the kink solution  $\phi(x, t) = \phi_k(x) + \psi(x, t)$ . We get for the function  $\psi(x, t)$  the following equation:

$$\psi_{xx} - \psi_{tt} - \gamma\psi_t + \frac{1}{2}(1 - 3\phi_k^2 + 2N(x))\psi = 0. \quad (4)$$

The study of the stability of the equilibrium solution  $\phi_k(x)$  leads to the following eigenvalue problem (we introduce  $\psi(x, t) = f(x) \exp(\lambda t)$  into Eq. (4)):

$$-f_{xx} + \frac{1}{2}(3\phi_k^2 - 1 - 2N(x))f = \Gamma f, \quad (5)$$

where  $\Gamma \equiv -\lambda^2 - \gamma\lambda$ .

For the functions  $F(x)$  and  $N(x)$  (defined as Eqs. (2) and (3)) the exact solution describing the static soliton can be written:  $\phi_k(x) = A \tanh(Bx)$ . The spectral problem (Eq. (5)) brings the following eigenvalues for the discrete spectrum:  $\Gamma_n = \frac{1}{2}A^2 - \frac{1}{2} + B^2(A + 2An - n^2 - 2)$ ; here  $A$  is defined as,  $A(A + 1) = (A^2/B^2) + 2$ . The integer part of  $A([A])$  defines the number of modes of the discrete spectrum.

The stability condition for the translational mode is,  $16B^4 + 2B^2(5 - 7A^2) + (1 - A^2)^2 < 0$ . When this condition is not fulfilled (thus the equilibrium point  $x=0$  is unstable) and  $A^2 > 1$ , then there will exist three equilibrium points for the soliton: two stable (at points  $x = x_1 > 0$  and  $x = x_2 < 0$ ) and one unstable at point  $x=0$ . This is because for huge values of  $|x|$  the leading inhomogeneity is  $F(x)$ , which is non-local and not zero at infinity. This inhomogeneity acts as a restoring force that pushes the soliton towards the point  $x=0$ . As a result of the competition between the local instability induced by  $N(x)\phi$  at point  $x=0$  and the non-local inhomogeneity  $F(x)$ , an effective double-well potential is created. This is equivalent to a pitchfork bifurcation.

We should make some remarks about the stability investigation. Writing down Eq. (4) we are making an approximation because the terms  $\psi^2$  and  $\psi^3$  are considered zero. Under this assumption the solutions of Eq. (4) can be used as an approximation for the kink dynamics only for small perturbation of the static soliton solution. However, the stability conditions obtained for the different modes are exact. In fact, when we say that the translational mode is stable for some set of values of the parameters, this means that in a neighborhood of this equilibrium point the effective potential for the soliton center of mass is a well (a minimum). On the contrary, when the parameters are changed such that the stability condition does not hold anymore, then a small deviation

in the initial condition of the soliton center of mass will cause the soliton to move away from the equilibrium position. The same is valid for the stability of the shape modes. For example, if the stability condition for the first shape mode is not satisfied, then small perturbation of the soliton profile will cause the soliton to explode. This has been checked numerically [11].

In general, the stability problems for perturbed soliton equations are very hard [9]. This is because in order to solve it, we first should have an exact solution of the equilibrium problem (which is rarely the case), and then one should solve the eigenvalue problem which usually has no solution in terms of elementary functions.

The investigation we have performed includes several steps. First, we have to solve an inverse problem in order to have external perturbations with the “shapes” that are relevant to the physical situations we want to discuss; second, we assure that the exact solutions will be known to us, and third, we should be able to solve exactly the stability problem. This last condition is fulfilled because Eq. (5) is a Schrödinger equation with a Pöschl–Teller potential [1–3]. The solution of this spectral problem can be found in Ref. [12].

In our case we were lucky enough to obtain exact solutions to perturbations that are generic and topologically equivalent to well-known perturbation models (e.g. the pitchfork bifurcation).

#### 4. Karhunen–Loève analysis

Let us consider a space-time-dependent force  $G(x, t)$  beside the space-dependent forces  $F(x)$  and  $N(x)\phi$ . In a previous work [1], González and Hołyst found that if  $G(x, t)$  has a spatial shape such that it coincides with one of the eigenfunctions of the stability operator of the soliton, then it is possible to get resonance if the frequency of the force also coincides with the resonant frequency of the considered mode. Therefore we can pump energy only into the translational mode of the kink selecting a space-time-dependent force of the form

$$G(x, t) = v \cos(\omega t) \left( \frac{1}{\cosh^A(B(x - x_1))} + \frac{1}{\cosh^A(B(x + x_1))} \right). \quad (6)$$

In Fig. 1a we present a sequence of bifurcations of the soliton center-of-mass coordinate  $X_{c.m.} = (\int_{-1/2}^{1/2} x \phi_x^2 dx) / (\int_{-1/2}^{1/2} \phi_x^2 dx)$  (sampled at times equal to multiples of the period of the driving force) as the driving amplitude  $v$  is increased and other parameters remain fixed ( $A = 1.22$ ,  $B = 0.32$ ,  $\omega = 1.22$ ,  $x_1 = 2.5$  and  $\gamma = 0.3$ ). For these values of  $A$  and  $B$  the stability condition for the translational mode is fulfilled, the soliton moves in a single-well potential and the system is in a regime with three discrete modes ( $[A] = 3$ ). Previous articles have studied the bistable case as well as the single-well case created by inhomogeneities of the type  $F(x)$  [1,2]. In this article we want to stress the complexity of the internal dynamics of the soliton when, besides  $F(x)$ , there is an impurity of the type  $N(x)\phi$ .

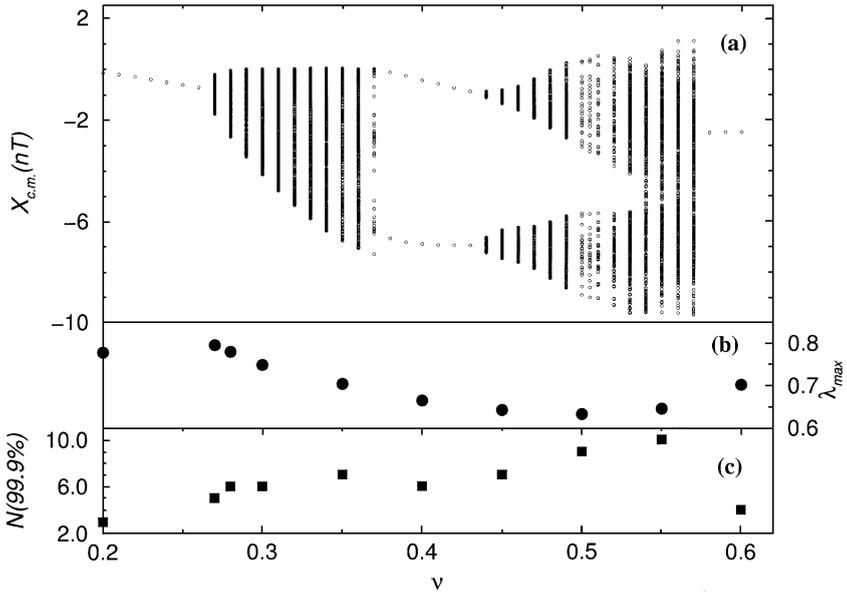


Fig. 1. (a) Bifurcation diagram for the position of the center of mass of the soliton. (b) Relative weight of the highest KL eigenvalue. (c) Number of KL modes that contains 99.9% of the dynamics.

We have integrated the equation using a standard implicit finite difference method with open boundary conditions  $\phi_x(0, t) = \phi_x(l, t) = 0$  and a system length  $l = 80$ . We use a kink-soliton with zero velocity as initial condition.

Poincaré maps for the soliton center-of-mass coordinate have revealed quasiperiodic and chaotic attractors for the non-periodic solutions of Fig. 1a: period one solutions precede a window of quasiperiodic bifurcations (the torus entangles as the amplitude of the time-dependent driving force increases). At a certain value a period two window appears and is followed by quasiperiodic (two-tori) bifurcations. For  $\nu = 0.55$  the Poincaré maps reveal high-dimensional chaotic motion followed by period one solutions.

The KL decomposition [13,14] allows to describe the dynamics in terms of an adequate basis of orthonormal functions or modes. The eigenvalues  $\lambda_n$  can be regarded as the weight of the mode  $n$ . Fig. 1b presents the greater eigenvalue normalized by the weight,  $W = \sum \lambda_n$ , whereas Fig. 1c presents the number of modes that contains 99.9% of the weight.

Fig. 2 reveals the increasing excitation of the KL modes as the amplitude of the space-time-dependent force increases. Note the sudden changes of the spectra when periodic motion is regained (period-two for  $\nu = 0.40$  and period-one for  $\nu = 0.60$ ). For these solutions the amplitude of the oscillations around the point  $x = 0$  diminishes even though the amplitude of the driving force has increased. This agrees with the higher contribution to the dynamics of the few modes of shape whereas all the rest of the modes decreased their contribution. Furthermore, for  $\nu = 0.60$  the first shape mode

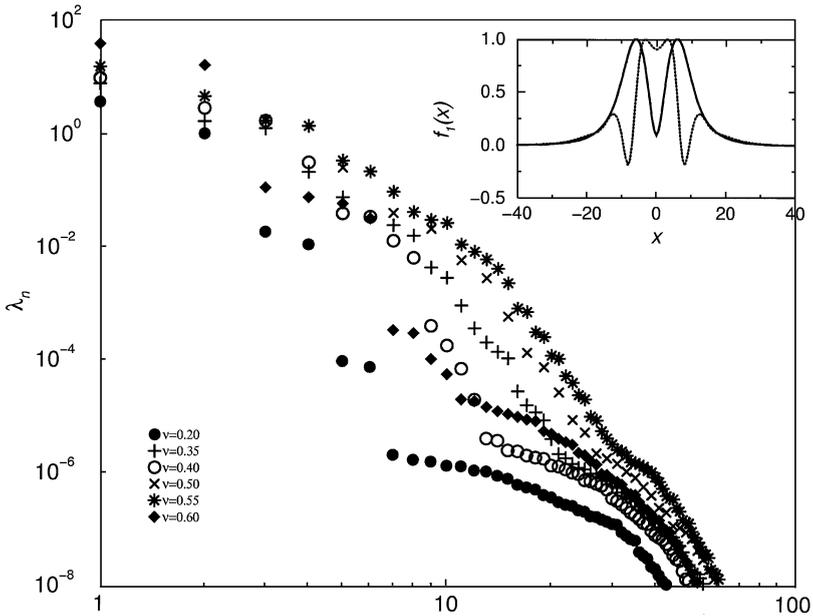


Fig. 2. KL spectra for the sequence of bifurcations presented in Fig. 1. The inset shows the first mode of the KL spectrum for  $\nu=0.20$  and  $\nu=0.60$ .

replaces the translational mode as the leading mode of the dynamics. The inset of the Fig. 2 presents the leading KL eigenmodes for the period-one solutions that initiates and ends the sequence of bifurcations considered in this section. The eigenmode for  $\nu=0.20$  appears to be the superposition of a pair of translational modes centered at the equilibrium points for the soliton. Similar situation occurs for  $\nu=0.60$  but the eigenvalue appears to be the superposition of a pair of shape modes.

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